We make use of the conditions of statics

$$T = 2\pi \int_0^a \tau_0(\rho)\, \rho d\rho, \qquad M = \int_0^a \int_0^{2\pi} (\sigma_1 e^{i\varphi} + \sigma_{-1} e^{-i\varphi})\, \rho^2 \cos\varphi\, d\rho\, d\varphi$$

After carrying out all the calculations, we obtain

$$T = 4\pi^2 \frac{a\theta}{\operatorname{sh} \pi\theta} \frac{\alpha A}{\gamma_1\gamma_2}, \qquad M = \frac{4\delta a^3\theta}{3H \operatorname{th} \pi\theta}(1+\theta^2) - \frac{4\pi^2 a^2\theta^2}{\operatorname{ch} \pi\theta} A \tag{3.3}$$

Expressions (3.2) and (3.3) enable us to determine the displacements of the punch

$$u_0 = \left[\frac{\pi\beta}{4a} + \frac{\gamma_1\gamma_2 H \operatorname{th} \pi\theta}{4a\theta(\theta^2+1)}(4\theta^2+1)\right] T + \frac{3H\alpha}{4a^2(\theta^2+1)} M$$

$$\delta = \frac{3H\alpha}{4a^3\theta(\theta^2+1)}\left[a\theta T + \frac{M}{\sqrt{\gamma_1\gamma_2}}\right] \tag{3.4}$$

In the case of an isotropic half-space the formulas (3.4) yield a solution which is in agreement with that given in [1].

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Translated by J. M. S.

EQUATIONS OF THE THEORY OF PERFECT PLASTICITY IN TERMS OF THE COMPONENTS OF DISPLACEMENT VELOCITIES

PMM Vol. 35, №1, 1971, pp. 183-185

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(Received August 5, 1968)

The equations of the theory of perfect plasticity are derived in terms of the components of displacement velocities. These equations are analogous to the Lamé equations in the theory of elasticity, when displacements are treated as the unknowns.

In the theory of perfect plasticity the components of stress can be expressed in terms of the components of displacement velocities by means of the following formulas [1, 2]:

$$\sigma_{ij} = \partial D / \partial e_{ij} \tag{1}$$

where $D = D(e_{ij})$ is the dissipation function, and e_{ij} are the components of the velocity of plastic deformation (for the sake of simplicity, the body is assumed rigidly plastic).

Substituting expressions (1) into the equations of equilibrium

$$\sigma_{ij,j} + F_i = 0 \tag{2}$$

we obtain

$$(\partial D/\partial e_{ij})_{,j} + F_i = 0 \tag{3}$$

If we now convert (3) to the components of displacement velocities u_i by means of the following formulas:

$$e_{ij} = {}^1\!/_2\,(u_{i,j} + u_{j,i}) \tag{4}$$

then the three equations (3) define the closed system of equations of the theory of perfect plasticity in terms of the three components of displacement velocities u_i.

These equations are analogous to the Lamé equations in the theory of elasticity.

Let us consider the case of an incompressible material. We shall designate the components of the deviator by a prime

$$e_{ij}' = e_{ij} - e\delta_{ij}, \qquad e = {}^1\!/_3 e_{kk} \tag{5}$$

The dissipation function can be written as $D = D\,(e'_{ij}, e)$. Formula (1) becomes

$$\sigma_{ij} = \frac{\partial D}{\partial e_{mn}'}\frac{\partial e_{mn}'}{\partial e_{ij}} + \frac{1}{3}\frac{\partial D}{\partial e}\delta_{ij}. \tag{6}$$

Using the notation $e_{11} = e_x, e_{12} = e_{xy}, \ldots$ we have

$$\frac{\partial e_x'}{\partial e_x} = \frac{2}{3}, \qquad \frac{\partial e_y'}{\partial e_x} = \frac{\partial e_z'}{\partial e_x} = -\frac{1}{3} \qquad (xyz) \tag{7}$$

The symbol (xyz) indicates that the expressions which have not been written out explicitly are obtainable by cyclic permutation of the indices. From (6) and (7) we have

$$\sigma_x = \frac{2}{3}\frac{\partial D}{\partial e_x'} - \frac{1}{3}\frac{\partial D}{\partial e_y'} - \frac{1}{3}\frac{\partial D}{\partial e_z'} + \frac{1}{3}\frac{\partial D}{\partial e}, \qquad \tau_{xy} = \frac{\partial D}{\partial e_{xy}} \qquad (xyz) \tag{8}$$

Summation of the first three expressions in (8) yields

$$\sigma = {}^1\!/_3\,(\sigma_x + \sigma_y + \sigma_z) = {}^1\!/_3\partial D/\partial e \tag{9}$$

Equation (8) takes now the form

$$\sigma_x' = \frac{2}{3}\frac{\partial D}{\partial e_x'} - \frac{1}{3}\frac{\partial D}{\partial e_y'} - \frac{1}{3}\frac{\partial D}{\partial e_z'}, \qquad \tau_{xy} = \frac{\partial D}{\partial e_{xy}} \qquad (x\,y\,z) \tag{10}$$

Formulas (10) relate the components of the deviator of stress to that of deformation. For an incompressible material $e = 0$ and the prime can be left out in components e_{ij}. Then, the final form of (10) for an incompressible material is

$$\sigma_x - \sigma = \frac{\partial D}{\partial e_x} - \frac{1}{3}\left(\frac{\partial D}{\partial e_x} + \frac{\partial D}{\partial e_y} + \frac{\partial D}{\partial e_z}\right), \qquad \tau_{xy} = \frac{\partial D}{\partial e_{xy}} \qquad (x\,y\,z) \tag{11}$$

The value of σ cannot be determined from (9) in the case of an incompressible material and remains indefinite, similarly as in the theory of elasticity.

Substitution of (11) into equilibrium equations (2) produces three equations with four unknowns: σ, u_x, u_y, u_z.

This system of equations is closed by the equation expressing the incompressibility which is

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{12}$$

It is a matter of common knowledge [2] that the edge of the dissipation function in space e_i corresponds to the boundary of the condition of plasticity in the space of principal stresses σ_i .

In this case, formulas (11) become of more general nature and take the following form:

$$\sigma_{ij}' = \sum_{k=1}^{m} \nu_k \left[\frac{\partial D_k}{\partial e_{ij}} - \frac{1}{3}\delta_{ij}\left(\frac{\partial D_k}{\partial e_{11}} + \frac{\partial D_k}{\partial e_{22}} + \frac{\partial D_k}{\partial e_{33}}\right)\right] \tag{13}$$

$$\nu_1 + \nu_2 + \ldots + \nu_m = 1$$

Let us consider actual expressions of the dissipation function under the conditions of plasticity of the maximum shear stress (condition of Tresca), maximum reduced stress and condition of von Mises.

In terms of the components of principal stresses, the plasticity condition of Tresca is $\sigma_i - \sigma_j \mid = 2k$.

In accordance with the associated flow rule, we have for the edge of the Tresca prism

$$e_i = \lambda, \quad e_j = -\lambda, \quad e_k = 0, \quad \lambda \geqslant 0$$

To the edge of the Tresca prism corresponds the boundary of the dissipation function, defined as the intersection of planes in the space of principal deformation velocities

$$D_1 = \sigma_i e_i = 2k\lambda = 2ke_i, \qquad D_2 = -2ke_j \tag{14}$$

The initial condition of plasticity is easily obtained from (14) and (13).

When (14) is rewritten in terms of tensor components e_{ij}, we can obtain from (13) and equilibrium equations (2) the initial equations expressed in terms of the components of displacement velocities.

We write the equations of the edge of Tresca prism as

$$\sigma_i - \sigma_j = \pm 2k, \qquad \sigma_i - \sigma_k = \pm 2k$$

Hence,

$$e_i = \lambda + \mu, \quad e_j = -\lambda, \quad e_k = -\mu$$

Our dissipation function has now the following form:

$$D = \sigma_i e_i = 2k(\lambda + \mu) = 2k\,|e_i|$$

Making use of the expressions for the second and third invariants of the deviator of deformation velocity J_2, J_3, we easily derive the equation defining our dissipation function, namely: $D^3 - 2k^2 J_2 D - 8k^3 J_3 = 0,\ J_2 = e_i^2 + e_j^2 + e_k^2,\ J_3 = e_i e_j e_k$ (15)

The condition of plasticity of the maximum reduced stress in terms of the components of principal stresses is of the form $2\sigma_i - \sigma_j - \sigma_k \cdot| = 2k$.

For the edge of the prism, the conditions of the maximum reduced stress are

$$e_i = 2\lambda, \quad e_j = -\lambda, \quad e_k = -\lambda, \quad \lambda \geqslant 0$$

To the boundary of the condition of plasticity of the maximum reduced stress corresponds the edge of the prism of the dissipation function which can be considered to be the intersection of the planes

$$D_1 = 2k\lambda = {}^2/_3 k\,(e_i - e_j), \qquad D_2 = {}^2/_3 k\,(e_i - e_k) \tag{16}$$

For the edge of the prism of maximum reduced stress

$$2\sigma_i - \sigma_j - \sigma_k = \pm 2k, \qquad 2\sigma_j - \sigma_k - \sigma_i = \pm 2k$$

we have

$$e_i = 2\lambda - \mu, \quad e_j = -\lambda + 2\mu, \quad e_k = -\lambda - \mu$$

The expression for the dissipation function is of the form $D = 2k\,|\,e_i + e_j\,|$.

It can be easily be shown that the dissipation function can be determined from

$$D^3 - 2k^2 J_2 D + 8k^3 J_3 = 0$$

Finally, for the von Mises condition of plasticity $\sigma_{ij}'\sigma_{ij}' = 2k^2$. it must be true that

$D = k\sqrt{e_{ij}e_{ij}}$.

In the case of plane deformation, all conditions of plasticity are reduced to a single condition $\sigma_1 - \sigma_2 = 2k$, $\sigma_1 > \sigma_2$. The dissipation function is then

$$D = k\sqrt{2e_{ij}e_{ij}} = k\sqrt{2}\,(e_x^2 + e_y^2 + 2e_{xy}^2)^{1/2} \qquad (17)$$

From (11) and (17) we obtain

$$\sigma_x' = ke_x\sqrt{2/J_2}, \quad \sigma_y' = ke_y\sqrt{2/J_2}, \quad \tau_{xy} = ke_{xy}\sqrt{2/J_2}. \qquad (18)$$

Substituting (18) into the equations of equilibrium and adding the equation expressing the incompressibility, i. e. $e_x + e_y = 0$, we obtain finally the equations which we intended to derive

Other particular cases can be investigated in an analogous manner.

The authors thank G. I. Bykovtsev for his valuable comments.

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Translated by J. M. S.

ON THE APPROXIMATE SOLUTION OF PROBLEMS OF LINEAR VISCOELASTICITY

PMM Vol. 35, №1, 1971, pp. 185-189

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(Received July 24, 1970)

An approximate solution of problems of linear viscoelasticity is derived. The method is applicable to both ageing and nonageing materials, as well as in numerical solution of related problems of elasticity. An estimate is made of the accuracy of the derived solution. The problem of a ponderable viscoelastic hemisphere lying on a horizontal smooth base is given as an example.

The solution of quasi-static problems of linear viscoelasticity for bodies with stationary boundaries reduces to the interpretation of the operator functions of viscoelasticity [1–3]. In the case of an isotropic material the viscoelastic properties are defined by two operators: $\mathbf{E}$ and $\boldsymbol{\nu}$. The dependence of the solution on operator $\mathbf{E}$, which can be determined by uncomplicated experiments on creep or relaxation, is simple. The dependence on operator $\boldsymbol{\nu}$ whose experimental determination is considerably more difficult is not negligible.

If the dependence of a solution on the Poisson ratio is complex, it is possible to obtain it by method of approximations [3–5].

1. Let us consider an arbitrary parameter of the stress-strain state $f(\mathbf{r}, \nu, t)$ of stressed elastic body whose dependence on time is determined by the variation of boundary conditions with time. Solution of the related problem of viscoelasticity is obtained by the substitution in the function f of operator $\boldsymbol{\nu}$ for the constant ν. The exact determination